

The jump of the Milnor number in the X_9 singularity class ^{*}

Szymon Brzostowski and Tadeusz Krasinski [†]

Faculty of Mathematics and Computer Science,
University of Łódź,
ul. Banacha 22, 90-238 Łódź, Poland

December 11, 2013

Abstract

The jump of the Milnor number of an isolated singularity f_0 is the minimal non-zero difference between the Milnor numbers of f_0 and one of its deformations (f_s) . We prove that for the singularities in the X_9 singularity class their jumps are equal to 2.

1 Introduction

Let $f_0 : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}, 0)$ be an *isolated singularity*, i.e. f_0 is a germ at 0 of a holomorphic function having an isolated critical point at $0 \in \mathbb{C}^n$, and $0 \in \mathbb{C}$ as the corresponding critical value. More specifically, there exists a representative $\hat{f}_0 : U \rightarrow \mathbb{C}$ of f_0 , holomorphic in an open neighborhood U of the point $0 \in \mathbb{C}^n$, such that $\hat{f}_0(0) = 0$, $\nabla \hat{f}_0(0) = 0$ and $\nabla \hat{f}_0(z) \neq 0$ for $z \in U \setminus \{0\}$, where for a holomorphic function f we put $\nabla f = \nabla_z f := (\partial f / \partial z_1, \dots, \partial f / \partial z_n)$.

In the sequel we will identify germs of holomorphic functions with their representatives or the corresponding convergent power series. The ring of germs of holomorphic functions of n variables will be denoted by \mathcal{O}^n .

A *deformation of the singularity* f_0 is the germ of a holomorphic function $f = f(s, z) : (\mathbb{C} \times \mathbb{C}^n, 0) \rightarrow (\mathbb{C}, 0)$ such that:

1. $f(0, z) = f_0(z)$,

^{*}**AMS subject classification:** 32S05, 14B05, 32S30, 14B07, **Keywords:** Milnor number, singularity, deformation of singularity

[†]*Email:* brzosts@math.uni.lodz.pl, *Email:* krasinsk@uni.lodz.pl

2. $f(s, 0) = 0$,
3. for each $|s| \ll 1$ it is $\nabla_z f(s, z) \neq 0$ for $z \neq 0$ in a (small) neighborhood of $0 \in \mathbb{C}^n$.

The deformation $f(s, z)$ of the singularity f_0 will also be treated as a family (f_s) of germs, taking $f_s(z) := f(s, z)$. In this context, the symbol ∇f_s will always denote $\nabla_z f_s$.

Remark. Notice that in the deformation (f_s) of f_0 there can occur *smooth* germs, that is germs satisfying $\nabla f_s(0) \neq 0$.

By the above assumptions it follows that, for every sufficiently small s , one can define a (finite) number μ_s as the Milnor number of f_s , namely

$$\mu_s := \mu(f_s) = \dim_{\mathbb{C}} \mathcal{O}^n / (\nabla f_s) = i_0 \left(\frac{\partial f_s}{\partial z_1}, \dots, \frac{\partial f_s}{\partial z_n} \right),$$

where the symbol $i_0 \left(\frac{\partial f_s}{\partial z_1}, \dots, \frac{\partial f_s}{\partial z_n} \right)$ denotes the multiplicity of the ideal $\left(\frac{\partial f_s}{\partial z_1}, \dots, \frac{\partial f_s}{\partial z_n} \right) \mathcal{O}^n$. Since the Milnor number is upper semi-continuous in the Zariski topology in families of singularities [GLS07, Ch. I, Thm. 2.6 and Ch. II, Prop. 2.57], there exists an open neighborhood S of the point $0 \in \mathbb{C}$ such that

1. $\mu_s = \text{const.}$ for $s \in S \setminus \{0\}$,
2. $\mu_0 \geq \mu_s$ for $s \in S$.

The (constant) difference $\mu_0 - \mu_s$ for $s \in S \setminus \{0\}$ will be called *the jump of the deformation* (f_s) and denoted by $\lambda((f_s))$. The smallest nonzero value among all the jumps of deformations of the singularity f_0 will be called *the jump (of the Milnor number) of the singularity* f_0 and denoted by $\lambda(f_0)$.

The first general result concerning the problem of computation of the jump was S. Gusein-Zade's [Gus93], who proved that there exist singularities f_0 for which $\lambda(f_0) > 1$ and that for irreducible plane curve singularities f_0 it holds $\lambda(f_0) = 1$. He showed that generic elements in some classes of singularities (satisfying conditions concerning the Milnor numbers and modality) fulfill $\lambda(f_0) > 1$, but he did not give any specific example of such a singularity.

The two-dimensional version of the problem of computation of the jump, and more precisely – of *the non-degenerate jump* (i.e. all the families (f_s) being considered are to be made of Kouchnirenko non-degenerate singularities), has been studied in [Bod07], [Wal10].

The following are examples of classes of singularities that fulfill the assumptions of the Gusein-Zade theorem.

1. The class X_9 , in the terminology of [AGV85]. It consists of singularities stably equivalent to the singularities of the form $f_0^a(x, y) := x^4 + y^4 + ax^2y^2, a \in \mathbb{C}, a^2 \neq 4$. The singularities are of modality 1 and $\mu(f_0^a) = 9$.

2. The class $W_{1,0}$, in the terminology of [AGV85]. It consists of singularities stably equivalent to the singularities of the form $f_0^{(a,b)}(x,y) := x^4 + y^6 + (a + by)x^2y^3, a, b \in \mathbb{C}, a^2 \neq 4$. The singularities are of modality 2 and $\mu(f_0^{(a,b)}) = 15$.

By the Gusein-Zade result, generic elements f of the classes X_9 and $W_{1,0}$ satisfy $\lambda(f) > 1$. However, determining the jump of any particular element of these classes is still an open and difficult problem. Gusein-Zade did not give any specific example of a singularity f with $\lambda(f) > 1$. The purpose of this work is to prove (Thm. 5) that for the singularities in the X_9 class we have

$$\lambda(f_0^a) = 2$$

(and that therefore all the singularities of the class X_9 are „generic” in the family X_9). In the class $W_{1,0}$ we obtain only a partial result (Prop. 3). Namely, for the singularities in $W_{1,0}$ that are stably equivalent to the ones in the subclass

$$f_0^{(0,b)}(x,y) = x^4 + y^6 + bx^2y^4, \quad b \in \mathbb{C},$$

we have

$$\lambda(f_0^{(0,b)}) = 1$$

(therefore these singularities are not „generic” in the family $W_{1,0}$).

This implies that the jump $\lambda(f_0)$ is not a topological invariant of singularities (Cor. 2).

In the light of the above results the following problems arise:

1. Show that for the remaining singularities in the $W_{1,0}$ class, i.e. for the singularities stably equivalent to $f_0^{(a,b)} := x^4 + y^6 + (a + by)x^2y^3$, where $a, b \in \mathbb{C}, 0 \neq a^2 \neq 4$, we have $\lambda(f_0^{(a,b)}) = 2$,

and more general ones (posed by Bodin in [Bod07]):

- (2) Find an algorithm that computes $\lambda(f_0)$.
- (3) Give the list of all possible Milnor numbers arising from deformations of f_0 (see [Wal10] for partial results in the non-degenerate case).

2 Preliminaries

Let \mathbb{N} be the set of nonnegative integers and \mathbb{R}_+ be the set of nonnegative real numbers. Let $f_0(x,y) = \sum_{(i,j) \in \mathbb{N}^2} a_{ij}x^i y^j$ be a singularity. Put $\text{supp}(f_0) := \{(i,j) \in \mathbb{N}^2 : a_{ij} \neq 0\}$. The *Newton diagram of f_0* is defined as the convex hull of the set

$$\bigcup_{(i,j) \in \text{supp}(f_0)} (i,j) + \mathbb{R}_+^2$$

and is denoted by $\Gamma_+(f_0)$. It is easy to see that the boundary (in \mathbb{R}^2) of the diagram $\Gamma_+(f_0)$ is a sum of two half-lines and a finite number of compact line segments. The set of those line segments will be called the *Newton polygon of the singularity f_0* and denoted by $\Gamma(f_0)$. For each segment $\gamma \in \Gamma(f_0)$ we define a weighted homogeneous polynomial

$$(f_0)_\gamma := \sum_{(i,j) \in \gamma} a_{ij} x^i y^j.$$

A singularity f_0 is called *non-degenerate (in the Kouchnirenko sense) on a segment $\gamma \in \Gamma(f_0)$* iff the system

$$\frac{\partial (f_0)_\gamma}{\partial x}(x,y) = 0, \frac{\partial (f_0)_\gamma}{\partial y}(x,y) = 0$$

has no solutions in $\mathbb{C}^* \times \mathbb{C}^*$. f_0 is called *non-degenerate* iff it is non-degenerate on every segment $\gamma \in \Gamma(f_0)$.

For the sake of simplicity, we consider the case of *convenient* singularities f_0 , i.e. we suppose that $\Gamma_+(f_0)$ intersects both coordinate axes in \mathbb{R}^2 . For such singularities we denote by S the area of the domain bounded by the coordinate axes and the Newton polygon $\Gamma(f_0)$. Let a (resp. b) be the distance of the point $(0,0)$ to the intersection of $\Gamma_+(f_0)$ with the horizontal (resp. vertical) axis. The number

$$v(f_0) := 2S - a - b + 1$$

is called the *Newton number of the singularity f_0* . Let us recall Planar Kouchnirenko Theorem.

Theorem 1. ([Kou76]) *For a convenient singularity f_0 we have:*

1. $\mu(f_0) \geq v(f_0)$,
2. if f_0 is non-degenerate then $\mu(f_0) = v(f_0)$.

Theorem 1 can be completed in the following way.

Theorem 2. (Płoski, [Plo90, Plo99]) *If for a convenient singularity f_0 there is $v(f_0) = \mu(f_0)$ then f_0 is non-degenerate.*

We will also need a „global” result concerning projective algebraic curves.

Theorem 3. ([GP01, Prop. 6.3]) *Let $\mathcal{C} \subset \mathbb{CP}^2$ be a projective algebraic curve of degree d . Suppose that m irreducible components of \mathcal{C} pass through a point $P \in \mathcal{C}$. Then the Milnor number $\mu_P(\mathcal{C})$ of \mathcal{C} at P satisfies the inequality*

$$\mu_P(\mathcal{C}) \leq (d-1)(d-2) + m - 1.$$

The rest of the section is devoted mainly to the concept of a versal unfolding. It is based on the book by Ebeling [Ebe07].

Let $f_0 : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}, 0)$ be a germ of a holomorphic function. An *unfolding* of f_0 is a holomorphic germ $F : (\mathbb{C}^n \times \mathbb{C}^k, 0) \rightarrow (\mathbb{C}, 0)$ such that $F(z, 0) = f_0(z)$ and $F(0, u) = 0$.

Two unfoldings $F : (\mathbb{C}^n \times \mathbb{C}^k, 0) \rightarrow (\mathbb{C}, 0)$ and $G : (\mathbb{C}^n \times \mathbb{C}^k, 0) \rightarrow (\mathbb{C}, 0)$ of f_0 are said to be *equivalent*, if there exists a holomorphic map-germ

$$\psi : (\mathbb{C}^n \times \mathbb{C}^k, 0) \rightarrow (\mathbb{C}^n, 0), \quad \psi(z, 0) = z, \quad \psi(0, u) = 0$$

such that

$$G(z, u) = F(\psi(z, u), u).$$

It is easy to see that this notion of equivalence is in fact an equivalence relation in the set of unfoldings of f_0 .

Let $F : (\mathbb{C}^n \times \mathbb{C}^k, 0) \rightarrow (\mathbb{C}, 0)$ be an unfolding of f_0 and $\phi : (\mathbb{C}^l, 0) \rightarrow (\mathbb{C}^k, 0)$ – a holomorphic map-germ. The *unfolding of f_0 induced from F by ϕ* is defined by the formula

$$G(z, u) = F(z, \phi(u)).$$

An unfolding $F : (\mathbb{C}^n \times \mathbb{C}^k, 0) \rightarrow (\mathbb{C}, 0)$ of f_0 is called *versal* if any unfolding of f_0 is equivalent to one induced from F .

The following proposition will be useful.

Proposition 1. ([Mar82, Ch. 4, Prop. 2.4]) *If $f \in \mathcal{O}^n$ is a singularity, \mathfrak{m} is the maximal ideal in \mathcal{O}^n , then*

$$\dim_{\mathbb{C}} \frac{\mathcal{O}^n}{\mathfrak{m}(\nabla f) \mathcal{O}^n} = \dim_{\mathbb{C}} \frac{\mathcal{O}^n}{(\nabla f) \mathcal{O}^n} + n.$$

The main result concerning versal unfoldings is the following.

Theorem 4. *Let $f_0 : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}, 0)$ be a singularity and put $\mu = \mu(f_0)$. Let $g_1, \dots, g_{\mu+n-1} \in \mathcal{O}^n$ be any representatives of a basis of the \mathbb{C} -vector space $\frac{\mathfrak{m}}{\mathfrak{m}(\nabla f_0)}$. Then the holomorphic germ*

$$F : (\mathbb{C}^n \times \mathbb{C}^{\mu+n-1}, 0) \rightarrow (\mathbb{C}, 0)$$

defined as

$$F(z, u) := u_1 g_1(z) + \dots + u_{\mu+n-1} g_{\mu+n-1}(z) + f_0(z)$$

is a versal unfolding of f_0 .

Remark. The proof of the above theorem runs in a very similar way to that given by Ebeling ([Ebe07, Prop. 3.17]); see also [Wal81, Thm. 3.4] for a more general, but less explicit, approach to the concept of a versal unfolding and a proof of Theorem 4.

Let $f : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}, 0)$ and $g : (\mathbb{C}^m, 0) \rightarrow (\mathbb{C}, 0)$ be two germs of holomorphic functions. We say that f is *stably equivalent* to g (see [AGV85]) iff there exists $p \in \mathbb{N}$, $p \geq \max(m, n)$, such that $\tilde{f} := f(z_1, \dots, z_n) + z_{n+1}^2 + \dots + z_p^2$ is biholomorphically

equivalent to $\tilde{g} := g(w_1, \dots, w_m) + w_{m+1}^2 + \dots + w_p^2$, i.e. there exists a biholomorphism $\Phi : (\mathbb{C}^p, 0) \rightarrow (\mathbb{C}^p, 0)$ such that $\tilde{f} \circ \Phi = \tilde{g}$.

It is easy to check that the Milnor number of a singularity is an invariant of the stable equivalence. The same is true for the jump of a singularity.

Proposition 2. *The jump of a singularity is an invariant of the stable equivalence.*

Proof. Since obviously $\lambda(f) = \lambda(g)$ for any two biholomorphically equivalent singularities f and g , it suffices to prove that for a singularity $f_0 : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}, 0)$ the equality

$$\lambda(f_0(z)) = \lambda(f_0(z) + z_{n+1}^2) \quad (1)$$

holds, where $z = (z_1, \dots, z_n)$.

First we consider the case $\mu(f_0) = 1$. Clearly, $\text{ord } f_0 = 2$. For the deformation $f_s(z) := f_0(z) + sz_1$ we have $\mu(f_0) - \mu(f_s) = 1$, $s \neq 0$. Hence $\lambda(f_0) = 1$. Similarly, $\lambda(f_0(z) + z_{n+1}^2) = 1$.

Now assume that $\mu(f_0) \geq 2$.

First note, that if (f_s) is a deformation of f_0 then the family $(f_s(z) + z_{n+1}^2)$ is a deformation of $f_0(z) + z_{n+1}^2$. Clearly, $\mu(f_s(z) + z_{n+1}^2) = \mu(f_s(z))$ so

$$\lambda(f_0(z)) \geq \lambda(f_0(z) + z_{n+1}^2).$$

To prove the opposite inequality we take a deformation (g_s) of $g_0(z, z_{n+1}) := f_0(z) + z_{n+1}^2$ that realizes $\lambda(g_0)$ i.e. $\mu(g_0) - \mu(g_s) = \lambda(g_0)$ for $s \neq 0$. Let, by Theorem 4, $h_1, \dots, h_{\mu+n-1} \in \mathcal{O}^n$ constitute a basis of $\frac{\mathfrak{m}_n}{\mathfrak{m}_n(\nabla f_0)\mathcal{O}^n}$, where $\mu := \mu(f_0)$ and \mathfrak{m}_n is the maximal ideal of \mathcal{O}^n . Then $h_1, \dots, h_{\mu+n-1}, z_{n+1}$ constitute a basis of $\frac{\mathfrak{m}_{n+1}}{\mathfrak{m}_{n+1}(\nabla g_0)\mathcal{O}^{n+1}}$. Hence, up to a biholomorphism, we may assume that

$$g_s(z, z_{n+1}) = v_1(s)h_1(z) + \dots + v_{\mu+n-1}(s)h_{\mu+n-1}(z) + v_{\mu+n}(s)z_{n+1} + f_0(z) + z_{n+1}^2,$$

for holomorphic $v_1, \dots, v_{\mu+n} : (\mathbb{C}, 0) \rightarrow (\mathbb{C}, 0)$.

We claim that the g_s 'es are not smooth. Indeed, in the opposite case we would have for $s \neq 0$

$$\lambda(g_0) = \mu(g_0) - \mu(g_s) = \mu(g_0). \quad (2)$$

On the other hand, for the deformation $\tilde{g}_s(z, z_{n+1}) := s(z_1^2 + \dots + z_n^2) + g_0(z, z_{n+1})$ of g_0 we would have, for sufficiently small $s \neq 0$, $\mu(\tilde{g}_s) = 1$ and then $\mu(g_0) - \mu(\tilde{g}_s) = \mu(f_0) - 1 > 0$. Hence $\lambda(g_0) \leq \mu(g_0) - \mu(\tilde{g}_s) = \mu(g_0) - 1$, a contradiction to (2).

Since the g_s 'es are not smooth, $v_{\mu+n} = 0$. Thus for the deformation

$$f_s(z) := v_1(s)h_1(z) + \dots + v_{\mu+n-1}(s)h_{\mu+n-1}(z) + f_0(z)$$

of f_0 there is $\mu(g_s) = \mu(f_s)$ and

$$\lambda(g_0) = \mu(g_0) - \mu(g_s) = \mu(f_0) - \mu(f_s) = \lambda(f_s).$$

This implies $\lambda(f_0) \leq \lambda(g_0)$. □

3 Main Results

In this section we will present the proofs of the results. We begin with the main theorem, concerning the class X_9 .

Theorem 5. *For the singularities*

$$f_0^a(x, y) = x^4 + y^4 + ax^2y^2, \quad (3)$$

where $a \in \mathbb{C}, a^2 \neq 4$, we have

$$\lambda(f_0^a) = 2.$$

Moreover, for every singularity of type X_9 its jump is equal to 2.

First we state and prove a lemma.

Lemma 1. *The (classes of the) monomials $x^i y^j$ with $0 < i + j \leq 3$ and the monomial $x^2 y^2$ form a basis of the \mathbb{C} -vector space $\mathfrak{m} / \mathfrak{m}(\nabla f_0^a)$.*

Proof. We have $\nabla f_0^a(x, y) = (4x^3 + 2axy^2, 4y^3 + 2ax^2y)$. Let us note that $x^5, x^3y \in \mathfrak{m}(\nabla f_0^a)$ because

$$x^5 = \left(\frac{x^2}{4} + \frac{2ay^2}{4(a^2 - 4)} \right) \frac{\partial f_0^a}{\partial x} + \left(\frac{-a^2xy}{4(a^2 - 4)} \right) \frac{\partial f_0^a}{\partial y}$$

and

$$x^3y = \left(\frac{-y}{(a^2 - 4)} \right) \frac{\partial f_0^a}{\partial x} + \left(\frac{ax}{2(a^2 - 4)} \right) \frac{\partial f_0^a}{\partial y}.$$

Since f_0^a is symmetric with respect to x and y , also $y^5, xy^3 \in \mathfrak{m}(\nabla f_0^a)$. Hence the classes of the monomials

$$x, y, x^2, xy, y^2, x^3, x^2y, xy^2, y^3, x^4, x^2y^2, y^4$$

generate $\mathfrak{m} / \mathfrak{m}(\nabla f_0^a)$. Since $x^4 \equiv -\frac{a}{2}x^2y^2$, $y^4 \equiv -\frac{a}{2}x^2y^2$ modulo $\mathfrak{m}(\nabla f_0^a)$, we get that the classes of the monomials $x^i y^j$ with $0 < i + j \leq 3$ and the monomial $x^2 y^2$ also generate the space $\mathfrak{m} / \mathfrak{m}(\nabla f_0^a)$. They form a basis of $\mathfrak{m} / \mathfrak{m}(\nabla f_0^a)$ because by Proposition 1 $\dim_{\mathbb{C}} \mathfrak{m} / \mathfrak{m}(\nabla f_0^a) = \dim_{\mathbb{C}} \mathcal{O}^n / \mathfrak{m}(\nabla f_0^a) - 1 = \dim_{\mathbb{C}} \mathcal{O}^n / (\nabla f_0^a) \mathcal{O}^n + 1 = \mu(f_0^a) + 1 = 10$. \square

Proof of Theorem 5. By Proposition 2 it is enough to prove the first part of the theorem. Let us fix $a \in \mathbb{C}, a^2 \neq 4$ and let $f_0 := f_0^a$. We have $\mu(f_0) = 9$. Let us consider the deformation

$$f_s(x, y) := x^4 + (y^2 + sx)^2 + ax^2(y^2 + sx)$$

of f_0 . Let us apply the change of coordinates: $x \mapsto x - sy^2$, $y \mapsto sy$, for $s \neq 0$. In these coordinates the f_s 's take the form

$$\bar{f}_s(x, y) = s^2x^2 + as^3xy^4 + s^4y^8 + [asx^3 + x^4 - 2as^2x^2y^2 - 4sx^3y^2 + 6s^2x^2y^4 - 4s^3xy^6].$$

It is easily seen that such \tilde{f}_s 'es are non-degenerate if $s \neq 0$. Thus, by Kouchnirenko theorem, we get $\mu(\tilde{f}_s) = \nu(\tilde{f}_s) = 7$ and so

$$\mu(f_s) = 7 \text{ for } s \neq 0. \quad (4)$$

This means that $\lambda((f_s)) = 2$ and therefore $\lambda(f_0) \leq 2$. By the definition of the jump of a singularity, there are only two cases: $\lambda(f_0) = 1$ or $\lambda(f_0) = 2$. We will exclude the first possibility. Suppose to the contrary, that there exists a deformation (f_s) of the singularity f_0 with the property that

$$\mu(f_s) = 8 \text{ for } s \neq 0. \quad (5)$$

By Theorem 4 and Lemma 1 we may assume that

$$f_s(x, y) = s_{10}(s)x + s_{01}(s)y + s_{20}(s)x^2 + s_{11}(s)xy + s_{02}(s)y^2 + s_{30}(s)x^3 + s_{21}(s)x^2y + s_{12}(s)xy^2 + s_{03}(s)y^3 + s_{22}(s)x^2y^2 + f_0(x, y),$$

where $s_{10}, \dots, s_{22} : (\mathbb{C}, 0) \rightarrow (\mathbb{C}, 0)$ are holomorphic. Since $\deg f_s = 4$ and $\mu(f_s) = 8$ for $s \neq 0$, by Theorem 3 three or four of the irreducible components of the curve $\mathcal{C}_s := \{(x, y) \in \mathbb{C}^2 : f_s(x, y) = 0\}$ pass through the origin. Hence $\text{ord } f_s = 3$ or $\text{ord } f_s = 4$, for $0 < |s| \ll 1$. The latter case is impossible by Theorem 1 because then $\mu(f_s) \geq \nu(f_s) \geq 9$. Thus, it suffices to consider the case $\text{ord } f_s = 3$. So, assume $\text{ord } f_s = 3$ for $s \neq 0$. Fix any small $s_0 \in \mathbb{C} \setminus \{0\}$. We can write

$$f_{s_0}(x, y) = s_{30}x^3 + s_{21}x^2y + s_{12}xy^2 + s_{03}y^3 + (s_{22} + a)x^2y^2 + x^4 + y^4,$$

with $s_{ij} = s_{ij}(s_0) \in \mathbb{C}$. Since $\text{ord } f_{s_0} = 3$, f_{s_0} has to be degenerate. Otherwise, by checking all the possible cases, we would get $\mu(f_{s_0}) \leq 6$ (by the Kouchnirenko theorem), which contradicts (5). Since $\gcd(3, 4) = 1$, the degeneracy of f_{s_0} may only happen on a segment of $\Gamma(f_{s_0})$ lying in the line: $u + v = 3$. So, we may write

$$f_{s_0}(x, y) = (\alpha x + \beta y)^2(\gamma x + \delta y) + (s_{22} + a)x^2y^2 + x^4 + y^4,$$

for some $\alpha, \beta, \gamma, \delta \in \mathbb{C}$, $|\alpha| + |\beta| > 0$, $|\gamma| + |\delta| > 0$. Moreover, $\alpha \neq 0$ and $\beta \neq 0$ because otherwise f_{s_0} would be non-degenerate. If we change coordinates: $\alpha x + \beta y \mapsto x$, $y \mapsto y$ then f_{s_0} takes the form

$$\tilde{f}_{s_0}(x, y) = x^2(\varepsilon x + \zeta y) + P_4(x, y),$$

where $\varepsilon, \zeta \in \mathbb{C}$, $|\varepsilon| + |\zeta| > 0$, and P_4 is a non-zero homogeneous polynomial of degree 4. We easily check, considering all the possible cases, that \tilde{f}_{s_0} is non-degenerate. So, again by the Kouchnirenko theorem, we would have

$$\mu(f_{s_0}) = \mu(\tilde{f}_{s_0}) = \nu(\tilde{f}_{s_0}) \leq 6,$$

which contradicts (5). □

Now we prove a partial result concerning the class $W_{1,0}$.

Proposition 3. For the singularities $f_0^{(0,b)}(x,y) = x^4 + y^6 + bx^2y^4$, where $b \in \mathbb{C}$, we have

$$\lambda(f_0^{(0,b)}) = 1.$$

In particular, $\lambda(x^4 + y^6) = 1$.

Proof. Fix $b \in \mathbb{C}$. Since $f_0^{(0,b)}$ is Kouchnirenko non-degenerate, it follows that $\mu(f_0^{(0,b)}) = \nu(f_0^{(0,b)}) = 15$. Consider the following deformation of $f_0^{(0,b)}$:

$$f_s^{(0,b)}(x,y) := x^4 + (y^2 + sx)^3 + bx^2y^4.$$

The deformation consists of degenerate singularities (for $s \neq 0$). Apply the following change of coordinates: $x \mapsto x - sy^2, y \mapsto sy$. In these coordinates the $f_s^{(0,b)}$ take the form

$$\tilde{f}_s^{(0,b)}(x,y) = s^3x^3 + (s^4 + bs^6)y^8 + [x^4 - 4sx^3y^2 + (6s^2 + bs^4)x^2y^4 - (4s^3 + 2bs^5)xy^6].$$

It is immediately seen that for $s \neq 0$ the singularities $\tilde{f}_s^{(0,b)}$ are non-degenerate and so

$$\mu(\tilde{f}_s^{(0,b)}) = 14.$$

Since the Milnor number is a biholomorphic (and even a topological) invariant of a singularity, there is also

$$\mu(f_s^{(0,b)}) = 14.$$

It means that for this particular deformation $(f_s^{(0,b)})$ of $f_0^{(0,b)}$ we have $\lambda((f_s^{(0,b)})) = 1$ and consequently $\lambda(f_0^{(0,b)}) = 1$. \square

Corollary 1. For every singularity f_0 stably equivalent to one of $f_0^{(0,b)}$, $b \in \mathbb{C}$, the jump $\lambda(f_0)$ of f_0 is equal to 1.

Proposition 3 implies also that $\lambda(f_0)$ is not a topological invariant of f_0 . Recall that two singularities f and g in \mathbb{C}^n have the same topological type if there exist neighbourhoods U and V of $0 \in \mathbb{C}^n$ and a homeomorphism $\Phi : U \rightarrow V$ such that $\Phi(V(f)) = V(g)$, where $V(f)$ (resp. $V(g)$) is the zero set of f (resp. g) in U (resp. V).

Corollary 2. The jump of the Milnor number $\lambda(f_0)$ is not a topological invariant of f_0 .

Proof. By the Gusein-Zade theorem, for generic elements $f_0^{(a,b)}$ of the class $W_{1,0}$ we have $\lambda(f_0^{(a,b)}) > 1$. Proposition 3 gives that the elements $f_0^{(0,b)}$ of $W_{1,0}$ satisfy $\lambda(f_0^{(0,b)}) = 1$. But all the singularities $f_0^{(a,b)}$, $a, b \in \mathbb{C}$, $a^2 \neq 4$, have the same topological type. This follows from the general Lê-Ramanujam theorem on μ -constant families of singularities or from the (much easier) fact that all the singularities $f_0^{(a,b)}$ have the same resolution graph. \square

Acknowledgement. We thank prof. A. Płoski for discussions which led to improvement of the text of the paper.

References

- [AGV85] Arnold, V. I., Gusein-Zade, S. M. and Varchenko, A. N. *Singularities of differentiable maps. Vol. I. The classification of critical points, caustics and wave fronts*, volume 82 of *Monographs in Mathematics*. Birkhäuser Boston Inc., Boston, MA, 1985.
- [Bod07] Bodin, A. Jump of Milnor numbers. *Bull. Braz. Math. Soc. (N.S.)*, 38(3):389–396, 2007.
- [Ebe07] Ebeling, W. *Functions of several complex variables and their singularities*, volume 83 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2007.
- [GLS07] Greuel, G.-M., Lossen, C. and Shustin, E. *Introduction to singularities and deformations*. Springer Monographs in Mathematics. Springer, Berlin, 2007.
- [GP01] Gwoździewicz, J. and Płoski, A. Formulae for the singularities at infinity of plane algebraic curves. *Univ. Iagel. Acta Math.*, (39):109–133, 2001. Effective methods in algebraic and analytic geometry (Kraków, 2000).
- [Gus93] Gusein-Zade, S. M. On singularities from which an A_1 can be split off. *Funct. Anal. Appl.*, 27(1):57–59, 1993.
- [Kou76] Kouchnirenko, A. G. Polyèdres de Newton et nombres de Milnor. *Invent. Math.*, 32(1):1–31, 1976.
- [Mar82] Martinet, J. *Singularities of smooth functions and maps*, volume 58 of *London Mathematical Society lecture note series*. Cambridge University Press, 1982.
- [Pło90] Płoski, A. Newton polygons and the Łojasiewicz exponent of a holomorphic mapping of C^2 . *Ann. Polon. Math.*, 51:275–281, 1990.
- [Pło99] Płoski, A. Milnor number of a plane curve and Newton polygons. *Univ. Iagel. Acta Math.*, 37:75–80, 1999. Effective methods in algebraic and analytic geometry (Bielsko-Biała, 1997).
- [Wal10] Walewska, J. The second jump of Milnor numbers. *Demonstratio Math.*, 43(2):361–374, 2010.
- [Wal81] Wall, C. T. C. Finite determinacy of smooth map-germs. *Bull. London Math. Soc.*, 13(6):481–539, 1981.